

From quadratic Hamiltonians of polymomenta to abstract geometrical Maxwell-like and Einstein-like equations

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Abstract

The aim of this paper is to create a large geometrical background on the dual 1-jet space $J^{1*}(\mathcal{T}, M)$ for a multi-time Hamiltonian approach of the electromagnetic and gravitational physical fields. Our geometric-physical construction is achieved starting only from a given quadratic Hamiltonian function

$$H = h_{ab}(t)g^{ij}(t, x)p_i^a p_j^b + U_{(a)}^{(i)}(t, x)p_i^a + \mathcal{F}(t, x)$$

which naturally produces a canonical nonlinear connection N , a canonical Cartan N -linear connection $CT(N)$ and their corresponding local distinguished (d-) torsions and curvatures. In such a context, we construct some geometrical electromagnetic-like and gravitational-like field theories which are characterized by some natural geometrical Maxwell-like and Einstein-like equations. Some abstract and geometrical conservation laws for the multi-time Hamiltonian gravitational physical field are also given.

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1 Distinguished Riemannian geometrization of metrical multi-time Hamilton spaces

Recently, the studies of Atanasiu and Neagu (see the papers [2], [3] and [4]) initiated the new way of distinguished Riemannian geometrization for Hamiltonians depending on polymomenta, which represents in fact a natural "multi-time" extension of the already classical Hamiltonian geometry on cotangent bundles (synthesized in the Miron et al.'s book [13]). In what follows, we expose the main geometrical ideas which characterize the distinguished Riemannian geometrical approach of Hamiltonians depending on polymomenta (see for details the Oană and Neagu's papers [15], [16]).

Let us consider that $h = (h_{ab}(t))$ is a semi-Riemannian metric on the "multi-time" (*temporal*) manifold \mathcal{T}^m , where $m = \dim \mathcal{T}$. Let $g = (g^{ij}(t^c, x^k, p_k^c))$ be a symmetric d-tensor on the dual 1-jet space $E^* = J^{1*}(\mathcal{T}, M)$, which has the rank $n = \dim M$ and a constant signature. At the same time, let us consider a smooth multi-time Hamiltonian function

$$E^* \ni (t^a, x^i, p_i^a) \rightarrow H(t^a, x^i, p_i^a) \in \mathbb{R},$$

which yields the *fundamental vertical metrical d-tensor*

$$G_{(a)(b)}^{(i)(j)} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i^a \partial p_j^b},$$

where $a, b = 1, \dots, m$ and $i, j = 1, \dots, n$.

Definition 1 A multi-time Hamiltonian function $H : E^* \rightarrow \mathbb{R}$, having the fundamental vertical metrical d-tensor of the form

$$G_{(a)(b)}^{(i)(j)}(t^c, x^k, p_k^c) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i^a \partial p_j^b} = h_{ab}(t^c) g^{ij}(t^c, x^k, p_k^c),$$

is called a **Kronecker h-regular multi-time Hamiltonian function**.

In this context, we introduce the following important geometrical concept:

Definition 2 A pair $MH_m^n = (E^* = J^{1*}(\mathcal{T}, M), H)$, where $m = \dim \mathcal{T}$ and $n = \dim M$, consisting of the dual 1-jet space and a Kronecker h-regular multi-time Hamiltonian function $H : E^* \rightarrow \mathbb{R}$, is called a **multi-time Hamilton space**.

Example 3 Let us consider the Kronecker h-regular multi-time Hamiltonian function $H_1 : E^* \rightarrow \mathbb{R}$, given by

$$H_1 = \frac{1}{4mc} h_{ab}(t) \varphi^{ij}(x) p_i^a p_j^b, \quad (1)$$

where $h_{ab}(t)$ ($\varphi_{ij}(x)$, respectively) is a semi-Riemannian metric on the temporal (spatial, respectively) manifold \mathcal{T} (M , respectively) having the physical meaning of **gravitational potentials**, and m and c are the known constants from Theoretical Physics representing the **mass of the test body** and the **speed of light**. Then, the multi-time Hamilton space $\mathcal{G}MH_m^n = (E^*, H_1)$ is called the **multi-time Hamilton space of the gravitational field**.

Example 4 If we consider on E^* a symmetric d-tensor field $g^{ij}(t, x)$, having the rank n and a constant signature, we can define the Kronecker h-regular multi-time Hamiltonian function $H_2 : E^* \rightarrow \mathbb{R}$, by setting

$$H_2 = h_{ab}(t) g^{ij}(t, x) p_i^a p_j^b + U_{(a)}^{(i)}(t, x) p_i^a + \mathcal{F}(t, x), \quad (2)$$

where $U_{(a)}^{(i)}(t, x)$ is a d -tensor field on E^* , and $\mathcal{F}(t, x)$ is a function on E^* . Then, the multi-time Hamilton space $\mathcal{NEDMH}_m^n = (E^*, H_2)$ is called the **non-autonomous multi-time Hamilton space of electrodynamics**. The dynamical character of the gravitational potentials $g_{ij}(t, x)$ (i.e., the dependence on the temporal coordinates t^c) motivated us to use the word "**non-autonomous**".

An important role for the subsequent development of our distinguished Riemannian geometrical theory for multi-time Hamilton spaces is represented by the following result (proved in the paper [2]):

Theorem 5 *If we have $m = \dim \mathcal{T} \geq 2$, then the following statements are equivalent:*

- (i) *H is a Kronecker h -regular multi-time Hamiltonian function on E^* .*
- (ii) *The multi-time Hamiltonian function H reduces to a multi-time Hamiltonian function of non-autonomous electrodynamic type. In other words, we have*

$$H = h_{ab}(t)g^{ij}(t, x)p_i^a p_j^b + U_{(a)}^{(i)}(t, x)p_i^a + \mathcal{F}(t, x). \quad (3)$$

Corollary 6 *The fundamental vertical metrical d -tensor of a Kronecker h -regular multi-time Hamiltonian function H has the form*

$$G_{(a)(b)}^{(i)(j)} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i^a \partial p_j^b} = \begin{cases} h_{11}(t)g^{ij}(t, x^k, p_k^1), & m = \dim \mathcal{T} = 1 \\ h_{ab}(t^c)g^{ij}(t^c, x^k), & m = \dim \mathcal{T} \geq 2. \end{cases} \quad (4)$$

Following now the Miron's geometrical ideas from [13], the paper [2] proves that any Kronecker h -regular multi-time Hamiltonian function H produces a natural nonlinear connection on the dual 1-jet space E^* , which depends only by the given Hamiltonian function H :

Theorem 7 *The pair of local functions $N = (N_{(i)b}^{(a)}, N_{(i)j}^{(a)})$ on E^* , where (χ_{bc}^a) are the Christoffel symbols of the semi-Riemannian temporal metric h_{ab})*

$$N_{(i)b}^{(a)} = \chi_{bc}^a p_i^c, \\ N_{(i)j}^{(a)} = \frac{h^{ab}}{4} \left[\frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial p_k^b} - \frac{\partial g_{ij}}{\partial p_k^b} \frac{\partial H}{\partial x^k} + g_{ik} \frac{\partial^2 H}{\partial x^j \partial p_k^b} + g_{jk} \frac{\partial^2 H}{\partial x^i \partial p_k^b} \right],$$

represents a nonlinear connection on E^ . This is called the **canonical nonlinear connection of the multi-time Hamilton space** $MH_m^n = (E^*, H)$.*

Taking into account the Theorem 5 and using the *generalized spatial Christoffel symbols* of the d -tensor g_{ij} , which are given by

$$\Gamma_{ij}^k = \frac{g^{kl}}{2} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

we immediately obtain the following geometrical result:

Corollary 8 For $m = \dim \mathcal{T} \geq 2$, the canonical nonlinear connection N of a multi-time Hamilton space $MH_m^n = (E^*, H)$, whose Hamiltonian function is given by (3), has the components

$$N_{(i)b}^{(a)} = \chi_{bc}^a p_i^c, \quad N_{(i)j}^{(a)} = -\Gamma_{ij}^k p_k^a + T_{(i)j}^{(a)},$$

where

$$T_{(i)j}^{(a)} = \frac{h^{ab}}{4} (U_{ib\bullet j} + U_{jb\bullet i}), \quad (5)$$

and

$$U_{ib} = g_{ik} U_{(b)}^{(k)}, \quad U_{kb\bullet r} = \frac{\partial U_{kb}}{\partial x^r} - U_{sb} \Gamma_{kr}^s.$$

The canonical nonlinear connection $N = \left(N_{(i)b}^{(a)}, N_{(i)j}^{(a)} \right)$ on E^* allows us the construction of the *adapted bases*

$$\left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a} \right\} \subset \chi(E^*), \quad \{ dt^a, dx^i, \delta p_i^a \} \subset \chi^*(E^*),$$

where

$$\begin{aligned} \frac{\delta}{\delta t^a} &= \frac{\partial}{\partial t^a} - N_{(j)a}^{(b)} \frac{\partial}{\partial p_j^b}, & \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(j)i}^{(b)} \frac{\partial}{\partial p_j^b}, \\ \delta p_i^a &= dp_i^a + N_{(i)b}^{(a)} dt^b + N_{(i)j}^{(a)} dx^j. \end{aligned} \quad (6)$$

The main result of the metrical multi-time Hamilton geometry is the Theorem of existence of the Cartan canonical h -normal N -linear connection $CT(N)$ (see [16]) which allows the subsequent development of our metrical multi-time Hamilton theory of physical fields.

Theorem 9 (the Cartan canonical N -linear connection) *On the metrical multi-time Hamilton space $MH_m^n = (J^{1*}(\mathcal{T}, M), H)$, endowed with the canonical nonlinear connection N , there exists an unique h -normal N -linear connection*

$$CT(N) = \left(\chi_{bc}^a, A_{jc}^i, H_{jk}^i, C_{j(c)}^{i(k)} \right),$$

having the metrical properties:

$$\begin{aligned} \text{(i)} \quad & g_{ij|k} = 0, \quad g^{ij}|_{(c)}^{(k)} = 0, \\ \text{(ii)} \quad & A_{jc}^i = \frac{g^{il}}{2} \frac{\delta g_{lj}}{\delta t^c}, \quad H_{jk}^i = H_{kj}^i, \quad C_{j(c)}^{i(k)} = C_{j(c)}^{k(i)}, \end{aligned}$$

where $"/_a$, $"/_k$ and $"/_{(c)}^{(k)}$ represent the local covariant derivatives of $CT(N)$. Moreover, the adapted local coefficients H_{jk}^i and $C_{j(c)}^{i(k)}$ of the **Cartan canonical connection** $CT(N)$ have the expressions

$$H_{jk}^i = \frac{g^{ir}}{2} \left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right), \quad C_{i(c)}^{j(k)} = -\frac{g_{ir}}{2} \left(\frac{\partial g^{jr}}{\partial p_k^c} + \frac{\partial g^{kr}}{\partial p_j^c} - \frac{\partial g^{jk}}{\partial p_r^c} \right).$$

Remark 10 (i) The Cartan canonical connection $CT(N)$ of the multi-time Hamilton space MH_m^n verifies also the metrical properties

$$h_{ab/c} = h_{ab|k} = h_{ab|_{(c)}}^{(k)} = 0, \quad g_{ij/c} = 0.$$

(ii) In the case $m = \dim \mathcal{T} \geq 2$, the adapted coefficients of the Cartan canonical connection $CT(N)$ of the multi-time Hamilton space MH_m^n reduce to

$$A_{bc}^a = \chi_{bc}^a, \quad A_{jc}^i = \frac{g^{il}}{2} \frac{\partial g_{lj}}{\partial t^c}, \quad H_{jk}^i = \Gamma_{jk}^i, \quad C_{j(c)}^{i(k)} = 0. \quad (7)$$

Applying the formulas that determine the local d-torsions and d-curvatures of an h -normal N -linear connection $D\Gamma(N)$ (see [15]) to the Cartan canonical connection $CT(N)$, we obtain (see [16]):

Theorem 11 The torsion tensor \mathbb{T} of the Cartan canonical connection $CT(N)$ of the multi-time Hamilton space MH_m^n is determined by the adapted local d-components

	$h_{\mathcal{T}}$	h_M		v	
	$m \geq 1$	$m = 1$	$m \geq 2$	$m = 1$	$m \geq 2$
$h_{\mathcal{T}} h_{\mathcal{T}}$	0	0	0	0	$R_{(r)ab}^{(f)}$
$h_M h_{\mathcal{T}}$	0	T_{1j}^r	T_{aj}^r	$R_{(r)1j}^{(1)}$	$R_{(r)aj}^{(f)}$
$vh_{\mathcal{T}}$	0	0	0	$P_{(r)1(1)}^{(1)(j)}$	$P_{(r)a(b)}^{(f)(j)}$
$h_M h_M$	0	0	0	$R_{(r)ij}^{(1)}$	$R_{(r)ij}^{(f)}$
vh_M	0	$P_{i(1)}^{r(j)}$	0	$P_{(r)i(1)}^{(1)(j)}$	0
vv	0	0	0	0	0

where

(i) for $m = \dim \mathcal{T} = 1$, we have

$$T_{1j}^r = -A_{j1}^r, \quad P_{i(1)}^{r(j)} = C_{i(1)}^{r(j)}, \quad P_{(r)1(1)}^{(1)(j)} = \frac{\partial N_{(r)1}^{(1)}}{\partial p_j^1} + A_{r1}^j - \delta_r^j \chi_{11}^1,$$

$$P_{(r)i(1)}^{(1)(j)} = \frac{\partial N_{(r)i}^{(1)}}{\partial p_j^1} + H_{ri}^j, \quad R_{(r)1j}^{(1)} = \frac{\delta N_{(r)1}^{(1)}}{\delta x^j} - \frac{\delta N_{(r)j}^{(1)}}{\delta t},$$

$$R_{(r)ij}^{(1)} = \frac{\delta N_{(r)i}^{(1)}}{\delta x^j} - \frac{\delta N_{(r)j}^{(1)}}{\delta x^i};$$

(ii) for $m = \dim \mathcal{T} \geq 2$, using the equality (5) and the notations

$$\chi_{fab}^c = \frac{\partial \chi_{fa}^c}{\partial t^b} - \frac{\partial \chi_{fb}^c}{\partial t^a} + \chi_{fa}^d \chi_{db}^c - \chi_{fb}^d \chi_{da}^c,$$

$$\mathfrak{R}_{kij}^r = \frac{\partial \Gamma_{ki}^r}{\partial x^j} - \frac{\partial \Gamma_{kj}^r}{\partial x^i} + \Gamma_{ki}^p \Gamma_{pj}^r - \Gamma_{kj}^p \Gamma_{pi}^r,$$

we have

$$\begin{aligned}
T_{aj}^r &= -A_{ja}^r, \quad P_{(r)a(b)}^{(f)(j)} = \delta_b^f A_{ra}^j, \quad R_{(r)ab}^{(f)} = \chi_{gab}^f p_r^g, \\
R_{(r)aj}^{(f)} &= -\frac{\partial N_{(r)j}^{(f)}}{\partial t^a} - \chi_{ca}^f T_{(r)j}^{(c)}, \\
R_{(r)ij}^{(f)} &= -\mathfrak{R}_{rij}^k p_k^f + \left[T_{(r)i|j}^{(f)} - T_{(r)j|i}^{(f)} \right].
\end{aligned}$$

Theorem 12 *The curvature tensor \mathbb{R} of the Cartan connection $CT(N)$ of the multi-time Hamilton space MH_m^n is determined by the following adapted local curvature d -tensors:*

	$h_{\mathcal{T}}$	h_M		v	
	$m \geq 1$	$m = 1$	$m \geq 2$	$m = 1$	$m \geq 2$
$h_{\mathcal{T}} h_{\mathcal{T}}$	χ_{abc}^d	0	R_{ibc}^l	0	$-R_{(l)(a)bc}^{(d)(i)}$
$h_M h_{\mathcal{T}}$	0	R_{i1k}^l	R_{ibk}^l	$-R_{(i)(1)1k}^{(1)(l)} = -R_{i1k}^l$	$-R_{(l)(a)bk}^{(d)(i)}$
$vh_{\mathcal{T}}$	0	$P_{i1(1)}^{l(k)}$	0	$-P_{(i)(1)1(1)}^{(1)(l)(k)} = -P_{i1(1)}^{l(k)}$	0
$h_M h_M$	0	R_{ijk}^l	\mathfrak{R}_{ijk}^l	$-R_{(i)(1)jk}^{(1)(l)} = -R_{ijk}^l$	$-R_{(l)(a)jk}^{(d)(i)}$
vh_M	0	$P_{ij(1)}^{l(k)}$	0	$-P_{(i)(1)j(1)}^{(1)(l)(k)} = -P_{ij(1)}^{l(k)}$	0
vv	0	$S_{i(1)(1)}^{l(j)(k)}$	0	$-S_{(i)(1)(1)(1)}^{(1)(l)(j)(k)} = -S_{i(1)(1)}^{l(j)(k)}$	0

where, for $m = \dim \mathcal{T} \geq 2$, we have the relations

$$-R_{(l)(a)bc}^{(d)(i)} = \delta_l^i \chi_{abc}^d - \delta_a^d R_{lbc}^i, \quad -R_{(l)(a)bk}^{(d)(i)} = -\delta_a^d R_{lbc}^i, \quad -R_{(i)(a)jk}^{(d)(l)} = -\delta_a^d \mathfrak{R}_{ijk}^l,$$

and, generally, the following formulas are true:

(i) for $m = \dim \mathcal{T} = 1$, we have

$$\begin{aligned}
\chi_{111}^1 &= 0, \\
R_{i1k}^l &= \frac{\delta A_{i1}^l}{\delta x^k} - \frac{\delta H_{ik}^l}{\delta t} + A_{i1}^r H_{rk}^l - H_{ik}^r A_{r1}^l + C_{i(1)}^{l(r)} R_{(r)1k}^{(1)}, \\
R_{ijk}^l &= \frac{\delta H_{ij}^l}{\delta x^k} - \frac{\delta H_{ik}^l}{\delta x^j} + H_{ij}^r H_{rk}^l - H_{ik}^r H_{rj}^l + C_{i(1)}^{l(r)} R_{(r)jk}^{(1)}, \\
P_{i1(1)}^{l(k)} &= \frac{\partial A_{i1}^l}{\partial p_k^1} - C_{i(1)/1}^{l(k)} + C_{i(1)}^{l(r)} P_{(r)1(1)}^{(1)(k)}, \\
P_{ij(1)}^{l(k)} &= \frac{\partial H_{ij}^l}{\partial p_k^1} - C_{i(1)|j}^{l(k)} + C_{i(1)}^{l(r)} P_{(r)j(1)}^{(1)(k)}, \\
S_{i(1)(1)}^{l(j)(k)} &= \frac{\partial C_{i(1)}^{l(j)}}{\partial p_k^1} - \frac{\partial C_{i(1)}^{l(k)}}{\partial p_j^1} + C_{i(1)}^{r(j)} C_{r(1)}^{l(k)} - C_{i(1)}^{r(k)} C_{r(1)}^{l(j)},
\end{aligned}$$

(ii) for $m = \dim \mathcal{T} \geq 2$, we have

$$\begin{aligned}\chi_{abc}^d &= \frac{\partial \chi_{ab}^d}{\partial t^c} - \frac{\partial \chi_{ac}^d}{\partial t^b} + \chi_{ab}^f \chi_{fc}^d - \chi_{ac}^f \chi_{fb}^d, \\ R_{ibc}^l &= \frac{\partial A_{ib}^l}{\partial t^c} - \frac{\partial A_{ic}^l}{\partial t^b} + A_{ib}^r A_{rc}^l - A_{ic}^r A_{rb}^l, \\ R_{ibk}^l &= \frac{\partial A_{ib}^l}{\partial x^k} - \frac{\partial \Gamma_{ik}^l}{\partial t^b} + A_{ib}^r \Gamma_{rk}^l - \Gamma_{ik}^r A_{rb}^l, \\ \mathfrak{R}_{ijk}^l &= \frac{\partial \Gamma_{ij}^l}{\partial x^k} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \Gamma_{ij}^r \Gamma_{rk}^l - \Gamma_{ik}^r \Gamma_{rj}^l.\end{aligned}$$

In the next Sections, following the physical and geometrical ideas of the already classical Lagrangian geometry of physical fields (see [12], [13] and [14]), we construct a possible multi-time Hamiltonian approach of the electromagnetic and gravitational physical fields, which is characterized by some natural geometrical Maxwell-like and Einstein-like equations. To reach this aim, we consider a multi-time Hamilton space $MH_m^n = (J^{1*}(T, M), H)$ endowed with its canonical nonlinear connection

$$N = \left(N_{1(i)b}^{(a)}, N_{2(i)j}^{(a)} \right),$$

and we also consider the Cartan canonical connection of the space MH_m^n , which is locally expressed by

$$CT(N) = \left(\chi_{bc}^a, A_{jc}^i, H_{jk}^i, C_{j(c)}^{i(k)} \right).$$

2 Multi-time Hamilton electromagnetism. Geometrical Maxwell-like equations

Let us consider the *canonical Liouville-Hamilton d-tensor field of polymomenta*

$$\mathbb{C}^* = p_i^a \frac{\partial}{\partial p_i^a},$$

together with the fundamental vertical metrical d-tensor $G_{(a)(b)}^{(i)(j)}$ of the multi-time Hamilton spaces MH_m^n . These geometrical objects allow us to construct the *metrical deflection d-tensors*

$$\begin{aligned}\Delta_{(a)b}^{(i)} &= G_{(a)(c)}^{(i)(k)} \Delta_{(k)b}^{(c)} = p_{(a)/b}^{(i)}, \quad \Delta_{(a)j}^{(i)} = G_{(a)(c)}^{(i)(k)} \Delta_{(k)j}^{(c)} = p_{(a)|j}^{(i)}, \\ \vartheta_{(a)(b)}^{(i)(j)} &= G_{(a)(c)}^{(i)(k)} \vartheta_{(k)(b)}^{(c)(j)} = p_{(a)|b}^{(i)(j)},\end{aligned}$$

where $p_{(a)}^{(i)} = G_{(a)(c)}^{(i)(k)} p_k^c$ and " /_b", " |_j" and " |_(b)" are the local covariant derivatives induced by the Cartan connection $CT(N)$.

Taking into account the expressions of the local covariant derivatives of the Cartan connection $CT(N)$ (see the paper [15]), by a direct calculation, we obtain

Proposition 13 *The metrical deflection d -tensors of the multi-time Hamilton spaces MH_m^n have the expressions:*

(i) *for $m = \dim \mathcal{T} = 1$, we have*

$$\begin{aligned}\Delta_{(1)1}^{(i)} &= -h_{11}g^{ik}A_{k1}^r p_r^1, & \Delta_{(1)j}^{(i)} &= h_{11}g^{ik} \left[-N_{2(k)j}^{(1)} - H_{kj}^r p_r^1 \right], \\ \vartheta_{(1)(1)}^{(i)(j)} &= h_{11}g^{ij} - h_{11}g^{ik}C_{k(1)}^{r(j)} p_r^1;\end{aligned}\tag{8}$$

(ii) *for $m = \dim \mathcal{T} \geq 2$, we have*

$$\begin{aligned}\Delta_{(a)b}^{(i)} &= -h_{ac}g^{ik}A_{kb}^r p_r^c, & \Delta_{(a)j}^{(i)} &= -\frac{g^{ik}}{4}(U_{ka\bullet j} + U_{ja\bullet k}), \\ \vartheta_{(a)(b)}^{(i)(j)} &= h_{ab}g^{ij}.\end{aligned}\tag{9}$$

In order to construct our metrical multi-time Hamiltonian theory of electromagnetism, we introduce the following concept:

Definition 14 *The distinguished 2-form on $J^{1*}(\mathcal{T}, M)$, locally defined by*

$$\mathbb{F} = F_{(a)j}^{(i)} \delta p_i^a \wedge dx^j + f_{(a)(b)}^{(i)(j)} \delta p_i^a \wedge \delta p_j^b,\tag{10}$$

where

$$F_{(a)j}^{(i)} = \frac{1}{2} \left[\Delta_{(a)j}^{(i)} - \Delta_{(a)i}^{(j)} \right], \quad f_{(a)(b)}^{(i)(j)} = \frac{1}{2} \left[\vartheta_{(a)(b)}^{(i)(j)} - \vartheta_{(a)(b)}^{(j)(i)} \right],\tag{11}$$

is called the **multi-time electromagnetic field of the metrical multi-time Hamilton space MH_m^n** .

By a straightforward calculation, the Proposition 13 implies

Proposition 15 *The components $F_{(a)j}^{(i)}$ and $f_{(a)(b)}^{(i)(j)}$ of the multi-time electromagnetic field \mathbb{F} , associated to the multi-time Hamilton space MH_m^n , have the following expressions:*

(i) *in the case $m = \dim \mathcal{T} = 1$, we have*

$$F_{(1)j}^{(i)} = \frac{h^{11}}{2} \left[g^{jk} N_{2(k)i}^{(1)} - g^{ik} N_{2(k)j}^{(1)} + (g^{jk} H_{ki}^r - g^{ik} H_{kj}^r) p_r^1 \right], \quad f_{(1)(1)}^{(i)(j)} = 0;$$

(ii) *in the case $m = \dim \mathcal{T} \geq 2$, we have*

$$F_{(a)j}^{(i)} = \frac{1}{8} \left[g^{jk} U_{ka\bullet i} - g^{ik} U_{ka\bullet j} + g^{jk} U_{ia\bullet k} - g^{ik} U_{ja\bullet k} \right], \quad f_{(a)(b)}^{(i)(j)} = 0.$$

The main result of our abstract geometrical Hamilton multi-time electromagnetism is given by

Theorem 16 *The electromagnetic components $F_{(a)j}^{(i)}$ of the multi-time Hamilton space MH_m^n are governed by the following **geometrical Maxwell-like equations**:*

(i) *for $m = \dim \mathcal{T} = 1$, we have*

$$\left\{ \begin{array}{l} F_{(1)k/1}^{(i)} = \frac{1}{2} \mathcal{A}_{\{i,k\}} \left\{ \Delta_{(1)1|k}^{(i)} + \Delta_{(1)r}^{(i)} T_{1k}^r + \vartheta_{(1)(1)}^{(i)(r)} R_{(r)1k}^{(1)} + R_{r1k}^i p_{(1)}^{(r)} \right\} \\ \sum_{\{i,j,k\}} F_{(1)j|k}^{(i)} = -\frac{1}{2} \sum_{\{i,j,k\}} \left\{ \vartheta_{(1)(1)}^{(i)(r)} R_{(r)jk}^{(1)} + R_{rjk}^i p_{(1)}^{(r)} \right\} \\ F_{(1)j|1}^{(i)} = \frac{1}{2} \mathcal{A}_{\{i,j\}} \left\{ \vartheta_{(1)(1)|j}^{(i)(k)} - P_{rj(1)}^i p_{(1)}^{(r)(k)} - \Delta_{(1)r}^{(i)} C_{j(1)}^{r(k)} - \vartheta_{(1)(1)}^{(i)(r)} P_{(r)j(1)}^{(1)(k)} \right\}; \end{array} \right.$$

(ii) *for $m = \dim \mathcal{T} \geq 2$, we have*

$$\left\{ \begin{array}{l} F_{(a)k/b}^{(i)} = \frac{1}{2} \mathcal{A}_{\{i,k\}} \left\{ \Delta_{(a)b|k}^{(i)} + \Delta_{(a)r}^{(i)} T_{bk}^r + \vartheta_{(a)(f)}^{(i)(r)} R_{(r)bk}^{(f)} + R_{rbk}^i p_{(a)}^{(r)} \right\} \\ \sum_{\{i,j,k\}} F_{(a)j|k}^{(i)} = -\frac{1}{2} \sum_{\{i,j,k\}} \left\{ \vartheta_{(a)(f)}^{(i)(r)} R_{(r)jk}^{(f)} + \mathfrak{R}_{rjk}^i p_{(a)}^{(r)} \right\} \\ \sum_{\{i,j,k\}} F_{(a)j|c}^{(i)} = 0, \end{array} \right.$$

where $\mathcal{A}_{\{i,j\}}$ means an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum, and we used the notations $p_{(1)}^{(i)} = G_{(1)(1)}^{(i)(j)} p_j^1$ and $p_{(a)}^{(i)} = G_{(a)(b)}^{(i)(j)} p_j^b$.

Proof. The general Ricci identities (see [15] and [16]) applied to g^{ij} lead us to the equalities:

$$\begin{aligned} g^{ir} R_{rbk}^j + g^{jr} R_{rbk}^i &= 0, & g^{ir} R_{rkl}^j + g^{jr} R_{rkl}^i &= 0, \\ g^{ir} P_{rk(c)}^j + g^{jr} P_{rk(c)}^i &= 0. \end{aligned} \quad (12)$$

Let us consider the following non-metrical deflection d-tensor identities ([15]):

$$\begin{aligned} (d_1) \quad & \Delta_{(p)b|k}^{(d)} - \Delta_{(p)k/b}^{(d)} = p_r^d R_{pbk}^r - \Delta_{(p)r}^{(d)} T_{bk}^r - \vartheta_{(p)(f)}^{(d)(r)} R_{(r)bk}^{(f)}, \\ (d_2) \quad & \Delta_{(p)j|k}^{(d)} - \Delta_{(p)k|j}^{(d)} = p_r^d R_{pj k}^r - \vartheta_{(p)(f)}^{(d)(r)} R_{(r)jk}^{(f)}, \\ (d_3) \quad & \Delta_{(p)j|c}^{(d)} - \vartheta_{(p)(c)|j}^{(d)(k)} = p_r^d P_{pj(c)}^r - \Delta_{(p)r}^{(d)} C_{j(c)}^{r(k)} - \vartheta_{(p)(f)}^{(d)(r)} P_{(r)j(c)}^{(f)(k)}, \end{aligned}$$

where $\Delta_{(i)b}^{(a)} = p_{i/b}^a$, $\Delta_{(i)j}^{(a)} = p_{i|j}^a$, $\vartheta_{(i)(b)}^{(a)(j)} = p_i^a |_{(b)}^{(j)}$.

Contracting the above deflection d-tensor identities with the fundamental vertical metrical d-tensor $G_{(a)(d)}^{(i)(p)}$, and using the equalities (12), we obtain the following *metrical deflection d-tensor identities*:

$$\begin{aligned} (d'_1) \quad & \Delta_{(a)b|k}^{(i)} - \Delta_{(a)k/b}^{(i)} = -p_{(a)}^{(r)} R_{rbk}^i - \Delta_{(a)r}^{(i)} T_{bk}^r - \vartheta_{(a)(f)}^{(i)(r)} R_{(r)bk}^{(f)}, \\ (d'_2) \quad & \Delta_{(a)j|k}^{(i)} - \Delta_{(a)k|j}^{(i)} = -p_{(a)}^{(r)} R_{rjk}^i - \vartheta_{(a)(f)}^{(i)(r)} R_{(r)jk}^{(f)}, \end{aligned}$$

$$(d'_3) \quad \Delta_{(a)j|^{(k)}_{(c)}}^{(i)} - \vartheta_{(a)(c)|j}^{(i)(k)} = -p_{(a)}^{(r)} F_{rj(c)}^i - \Delta_{(a)r}^{(i)} C_{j(c)}^{r(k)} - \vartheta_{(a)(f)}^{(i)(r)} P_{(r)j(c)}^{(f)(k)}.$$

To obtain the first (respectively, the third) geometrical Maxwell-like equation, we permute the indices i and k in the identity (d'_1) (respectively, the indices i and j in the identity (d'_3)), and we subtract this new identity from the initial one. For $m = \dim \mathcal{T} \geq 2$ we find the following new identity:

$$F_{(a)j|^{(k)}_{(c)}}^{(i)} = \frac{1}{2} \left[\vartheta_{(a)(c)|j}^{(i)(k)} - \vartheta_{(a)(c)|i}^{(j)(k)} \right].$$

Consequently, doing a cyclic sum by $\{i, j, k\}$ for $m \geq 2$, we obtain what we were looking for.

Doing a cyclic sum after the indices $\{i, j, k\}$ in the identity (d'_2) , it follows the second geometrical Maxwell-like equation. ■

3 Multi-time Hamilton gravitational field. Geometrical Einstein-like equations

Let us consider that $h = (h_{ab}(t))$ is a fixed semi-Riemannian metric on the temporal manifold \mathcal{T} and let

$$N = \left(N_{(i)b}^{(a)}, N_{(i)j}^{(a)} \right)$$

be an "a priori" given nonlinear connection on the dual 1-jet space $J^{1*}(\mathcal{T}, M)$. Let

$$\delta p_i^a = dp_i^a + N_{(i)b}^{(a)} dt^b + N_{(i)j}^{(a)} dx^j$$

be the vertical distinguished 1-forms adapted to the nonlinear connection N .

An essential element in the development of our abstract geometrical multi-time Hamilton gravitational theory is given by the following definition:

Definition 17 *From an abstract physical point of view, an adapted metrical d-tensor \mathbb{G} on the dual 1-jet space $E^* = J^{1*}(\mathcal{T}, M)$, locally expressed by*

$$\mathbb{G} = h_{ab} dt^a \otimes dt^b + g_{ij} dx^i \otimes dx^j + h_{ab} g^{ij} \delta p_i^a \otimes \delta p_j^b,$$

where $g_{ij} = g_{ij}(t^c, x^k, x_c^k)$ is a symmetric d-tensor field of rank $n = \dim M$ having a constant signature on $E^* = J^{1*}(\mathcal{T}, M)$, is called a **multi-time gravitational h-potential** on E^* .

Now, taking a multi-time Hamilton space $MH_m^n = (E^*, H)$, via its fundamental vertical metrical d-tensor $G_{(a)(b)}^{(i)(j)}$ (which is given by (4)) and its canonical nonlinear connection N , we naturally construct a corresponding multi-time gravitational h-potential on E^* , setting

$$\mathbb{G} = h_{ab} dt^a \otimes dt^b + g_{ij} dx^i \otimes dx^j + h_{ab} g^{ij} \delta p_i^a \otimes \delta p_j^b.$$

At the same time, let us consider that

$$CT(N) = \left(\chi_{ab}^c, A_{jc}^k, H_{jk}^i, C_{j(c)}^{i(k)} \right)$$

is the Cartan canonical connection of the multi-time Hamilton space MH_m^n .

Postulate. *We postulate that the **geometrical Einstein-like equations**, which govern the multi-time gravitational h-potential \mathbb{G} of the multi-time Hamilton space MH_m^n , are the abstract geometrical Einstein equations attached to the Cartan canonical connection $CT(N)$ and to the adapted metric \mathbb{G} on E^* , namely*

$$\text{Ric}(CT) - \frac{\text{Sc}(CT)}{2}\mathbb{G} = \mathcal{K}\mathbb{T}, \quad (13)$$

where $\text{Ric}(CT)$ represents the **Ricci tensor** of the Cartan connection, $\text{Sc}(CT)$ is the **scalar curvature**, \mathcal{K} is the **Einstein constant** and \mathbb{T} is an intrinsic d-tensor of matter, which is called the **stress-energy d-tensor of polymomenta**.

In the adapted basis of vector fields

$$(X_A) = \left(\frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a} \right),$$

which is produced by the canonical nonlinear connection N of the multi-time Hamilton space MH_m^n , the curvature tensor \mathbb{R} of the Cartan canonical connection $CT(N)$ is locally expressed by

$$\mathbb{R}(X_C, X_B)X_A = \mathbf{R}_{ABC}^D X_D.$$

It follows that we have

$$R_{AB} = \text{Ric}(X_A, X_B) = \mathbf{R}_{ABD}^D, \quad \text{Sc}(CT) = G^{AB}R_{AB},$$

where

$$G^{AB} = \begin{cases} h^{ab}, & \text{for } A = a, B = b \\ g^{ij}, & \text{for } A = i, B = j \\ h^{ab}g_{ij}, & \text{for } A = \binom{a}{i}, B = \binom{b}{j} \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

Taking into account, on the one hand, the form of the metrical d-tensor $\mathbb{G} = (G_{AB})$ of the multi-time Hamilton space MH_m^n , and, on the other hand, taking into account the expressions of the local curvature d-tensors attached to the Cartan canonical connection $CT(N)$, by direct computations, we find

Proposition 18 *The Ricci tensor $\text{Ric}(CT)$ of the Cartan canonical connection $CT(N)$ of the multi-time Hamilton space MH_m^n is determined by the following adapted components:*

(i) for $m = \dim \mathcal{T} = 1$, we have

$$\begin{aligned} R_{11} &:= \chi_{11} = 0, & R_{1i} &= R_{1i1}^1 = 0, \\ R_{1(1)}^{(i)} &= -P_{1(1)1}^{1(i)} = 0, & R_{i1} &= R_{i1r}^r, & R_{ij} &= R_{ijr}^r, \\ R_{(1)1}^{(i)} &:= -P_{(1)1}^{(i)} = -P_{r1(1)}^{i(r)}, & R_{i(1)}^{(j)} &:= -P_{i(1)}^{(j)} = -P_{ir(1)}^{r(j)}, \\ R_{(1)(1)}^{(i)(j)} &:= -S_{(1)(1)}^{(i)(j)} = -S_{r(1)(1)}^{i(j)(r)}, & R_{(1)j}^{(i)} &:= -P_{(1)j}^{(i)} = -P_{rj(1)}^{i(r)}; \end{aligned}$$

(ii) for $m = \dim \mathcal{T} \geq 2$, we have

$$\begin{aligned} R_{ab} &:= \chi_{ab} = \chi_{abf}^f, & R_{ai} &= R_{aif}^f = 0, \\ R_{a(b)}^{(j)} &= -P_{af(b)}^{f(j)} = 0, & R_{ia} &= R_{iar}^r, & R_{ij} &= R_{ijr}^r, \\ R_{(a)(b)}^{(i)(j)} &:= -S_{(a)(b)}^{(i)(j)} = -S_{r(a)(b)}^{i(j)(r)} = 0, & R_{i(b)}^{(j)} &:= -P_{i(b)}^{(j)} = -P_{ir(b)}^{r(j)} = 0, \\ R_{(a)b}^{(i)} &:= -P_{(a)b}^{(i)} = -P_{rb(a)}^{i(r)} = 0, & R_{(a)j}^{(i)} &:= -P_{(a)j}^{(i)} = -P_{rj(a)}^{i(r)} = 0. \end{aligned}$$

Using the notations $\chi = h^{ab}\chi_{ab}$, $R = g^{ij}R_{ij}$ and $S = h^{11}g_{ij}S_{(1)(1)}^{(i)(j)}$, we obtain

Corollary 19 *The scalar curvature $\text{Sc}(\text{CT})$ of the Cartan canonical connection $\text{CT}(N)$ of the multi-time Hamilton space MH_m^n is given by the formulas:*

- (i) for $m = \dim \mathcal{T} = 1$, we have $\text{Sc}(\text{CT}) = R - S$;
- (ii) for $m = \dim \mathcal{T} \geq 2$, we have $\text{Sc}(\text{CT}) = \chi + R$.

In this context, the main result of the Hamilton geometrical multi-time gravitational theory is offered by

Theorem 20 *The **geometrical Einstein-like equations**, which govern the multi-time gravitational h -potential \mathbb{G} of the multi-time Hamilton space MH_m^n , have the following adapted local form:*

(i) for $m = \dim \mathcal{T} = 1$, we have

$$\left\{ \begin{aligned} & -\frac{R-S}{2}h_{11} = \mathcal{K}\mathbb{T}_{11} \\ & R_{ij} - \frac{R-S}{2}g_{ij} = \mathcal{K}\mathbb{T}_{ij} \\ & -S_{(1)(1)}^{(i)(j)} - \frac{R-S}{2}h_{11}g^{ij} = \mathcal{K}\mathbb{T}_{(1)(1)}^{(i)(j)} \\ & 0 = \mathbb{T}_{1i}, \quad R_{i1} = \mathcal{K}\mathbb{T}_{i1} \\ & 0 = \mathbb{T}_{1(1)}^{(i)}, \quad -P_{i(1)}^{(j)} = \mathcal{K}\mathbb{T}_{i(1)}^{(j)} \\ & -P_{(1)1}^{(i)} = \mathcal{K}\mathbb{T}_{(1)1}^{(i)}, \quad -P_{(1)j}^{(i)} = \mathcal{K}\mathbb{T}_{(1)j}^{(i)}; \end{aligned} \right. \quad (15)$$

(ii) for $m = \dim \mathcal{T} \geq 2$, we have

$$\left\{ \begin{array}{l} \chi_{ab} - \frac{\chi + R}{2} h_{ab} = \mathcal{K} \mathbb{T}_{ab} \\ R_{ij} - \frac{\chi + R}{2} g_{ij} = \mathcal{K} \mathbb{T}_{ij} \\ -\frac{\chi + R}{2} h_{ab} g^{ij} = \mathcal{K} \mathbb{T}_{(a)(b)}^{(i)(j)} \\ 0 = \mathbb{T}_{ai}, \quad R_{ia} = \mathcal{K} \mathbb{T}_{ia} \\ 0 = \mathbb{T}_{(a)b}^{(i)}, \quad 0 = \mathbb{T}_{a(b)}^{(j)} \\ 0 = \mathbb{T}_{i(b)}^{(j)}, \quad 0 = \mathbb{T}_{(a)j}^{(i)}, \end{array} \right. \quad (16)$$

where \mathbb{T}_{AB} , $A, B \in \left\{ a, i, \begin{smallmatrix} (a) \\ (i) \end{smallmatrix} \right\}$ represent the adapted components of the stress-energy d-tensor of matter \mathbb{T} .

Remark 21 In order to have the compatibility of the system of geometrical Einstein-like equations, it is necessary the "a priori" vanishing of certain adapted components of the stress-energy d-tensor of matter \mathbb{T} .

From a physical point of view, it is well known that in the classical Riemannian theory of gravity, the stress-energy d-tensor of matter have to verify some conservation laws. By a natural extension of the classical Riemannian conservation laws, in our geometrical Hamiltonian context, we postulate the following *generalized conservation laws* of the stress-energy d-tensor of polymomenta \mathbb{T} :

$$\mathbb{T}_{A|B}^B = 0, \quad \forall A \in \left\{ a, i, \begin{smallmatrix} (a) \\ (i) \end{smallmatrix} \right\},$$

where $\mathbb{T}_A^B = G^{BD} \mathbb{T}_{DA}$. Consequently, by straightforward computations, we obtain

Theorem 22 The *generalized conservation laws* of the Einstein-like equations of the multi-time Hamilton space MH_m^n are expressed by the following formulas:

(i) for $m = \dim \mathcal{T} = 1$, we have

$$\left\{ \begin{array}{l} \left[\frac{R - S}{2} \right]_{/1} = R_{1|r}^r - P_{(r)1}^{(1)} |_{(1)}^{(r)} \\ \left[R_j^r - \frac{R - S}{2} \delta_j^r \right]_{|r} = P_{(r)j}^{(1)} |_{(1)}^{(r)} \\ \left[S_{(r)(1)}^{(1)(j)} + \frac{R - S}{2} \delta_r^j \right] |_{(1)}^{(r)} = -P_{(1)|r}^{r(j)}, \end{array} \right. \quad (17)$$

where

$$\begin{aligned}
R_1^i &= g^{iq} R_{q1}, & P_{(i)1}^{(1)} &= h^{11} g_{iq} P_{(1)1}^{(q)}, & R_j^i &= g^{iq} R_{qj}, \\
P_{(i)j}^{(1)} &= h^{11} g_{iq} P_{(1)j}^{(q)}, & P_{(1)}^{i(j)} &= g^{iq} P_{q(1)}^{(j)} & S_{(i)(1)}^{(1)(j)} &= h^{11} g_{iq} S_{(1)(1)}^{(q)(j)};
\end{aligned}$$

(ii) for $m = \dim \mathcal{T} \geq 2$, we have

$$\left\{ \begin{array}{l} \left[\chi_b^f - \frac{\chi + R}{2} \delta_b^f \right]_{/f} = -R_{b|r}^r \\ \left[R_j^r - \frac{\chi + R}{2} \delta_j^r \right]_{|r} = 0, \end{array} \right. \quad (18)$$

where

$$\chi_b^f = h^{fc} \chi_{cb}, \quad R_j^i = g^{iq} R_{qj}, \quad R_b^i = g^{iq} R_{qb}.$$

Open problem. The authors of this paper consider that the finding of a possible real physical meaning of the present multi-time Hamiltonian geometrical theories of gravity and electromagnetism may be an open problem for physicists.

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